Definitions and key facts for section 3.3

For any $n \times n$ matrix and $\mathbf{b}$ in $\mathbb{R}^n$, let $A_j(\mathbf{b})$ be the matrix obtained by replacing the jth of A with $\mathbf{b}$.

$$A_j(\mathbf{b}) = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{j-1} & \mathbf{b} & \mathbf{a}_{j+1} & \cdots & \mathbf{a}_n \end{bmatrix}.$$

Fact: Cramer's rule
If A is an invertible $n \times n$ matrix, then for any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A} \qquad \text{for } i = 1, 2, \ldots, n.$$

From this we can obtain a *formula* for the inverse of A. As it involves determinants, it is wildly impractical for calculations but it is a powerful tool to have for theoretical work. Using Cramer's rule, one can see that the (i, j) entry of A^{-1} is (j, i)-cofactor C_{ji} divided by $\det A$:

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Fact: The geometry of the determinant If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelopiped determined by the columns of A is $|\det A|$.

Fact: The determinant and linear transformations If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with standard matrix A and S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}.$$

Similarly, if $T : \mathbb{R}^3 \to \mathbb{R}^3$ with standard matrix A and S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}.$$

In fact, this can be extended to any region S in $\mathbb{R}^2$ ($\mathbb{R}^3$) with finite area (volume).